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Natural convection of viscoelastic fluids in a vertical slot

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Linear stability theory is applied to the natural convection of slightly elastic, viscous fluids in an infinitely long vertical slot. Travelling, as well as stationary, disturbances are considered. It is found that the elasticity (i) slightly stabilizes the stationary disturbances while strongly destabilizing the travelling disturbances, (ii) strongly increases the wave speed while slightly decreasing the wavenumber and (iii) reduces the transition Prandtl number, which separates the stationary cells from the travelling waves, from its value of 12·7 for Newtonian fluids.

Experiments are carried out with a viscoelastic fluid prepared by mixing Separan AP30 with water, giving a Prandtl number of about 30. This fluid is shown to produce wave instability at a Grashof number between 3900 and 4300. Under the same conditions, a Newtonian fluid is shown to remain stable to both stationary and travelling disturbances.

1. Introduction

In the past decade, considerable work has been done on the isothermal stability of viscoelastic fluids. Yet, studies on the thermal stability of these fluids are recent and rather few, and are restricted to problems related to that of Bénard. The purpose of this investigation is to consider the effect of elasticity on the stability of natural convection in a vertical slot. Since the stability of the Bénard and slot problems differ in many ways, the study of the latter is expected to increase our understanding of the thermal stability of viscoelastic fluids.

Two special slot problems frequently considered for Newtonain fluids assume either uniform of linearly varying plate temperatures. The first problem (with uniform plate temperatures) has, depending on the parameter ranges, two solutions, which are called the conduction and convection regimes. The primary flow of the convection regime, although well known qualitatively, has not been analytically studied to the degree that may be necessary to determine its stability. On the other hand, the second problem (with linearly varying plate temperatures), although it has a unique solution, is somewhat difficult to realize experimentally. In view of these facts, it was decided to concentrate on the analytical and qualitative experimental aspects of the conduction regime of the first problem.

The literature on the thermal stability of Newtonian fluids in a slot is quite extensive, and will not be reviewed here. For a complete list of references the reader is referred to the article by Hart (1971). The original work on viscoelastic fluids appears to be that of Herbert (1963) on plane Couette flow heated from below; he found a finite elastic stress in the undisturbed state to be required for elasticity to affect the stability (this, of course, should be expected in view of the fact that he was only interested in stationary disturbances); using a three-constant rheological model due to Oldroyd (1950) he demonstrated, for finite rates of strain, that the elasticity has a destabilizing effect which results solely from the change in apparent viscosity. In another study, Green (1968) reported, for free boundaries, the overstability of the Bénard problem in terms of a two-constant model due to Oldroyd. The same problem in terms of the Maxwell model, including rigid boundaries as well and the effect of elasticity on the variational principle, was investigated by Vest & Arpaci (1969 *b*). Both studies expected and found elasticity to have no effect on stationary disturbances but a large effect on travelling disturbances. The latest work on the stability of the viscoelastic Bénard problem is that of Sokolov & Tanner (1972), who tried, after some general considerations, to consolidate their results with those of Green (1968) and Vest & Arpaci (1969*b*).

Literature on the isothermal stability of viscoelastic fluids becomes relevant to the thermal stability of these fluids only during the process of selecting the appropriate rheological models. Consequently, excluding a few problems related to boundary layers, to inclined surfaces and to Couette flow, we concentrate on the literature associated with the Taylor problem, which has received by far the most attention in view of its simplicity and its relevance to viscometry. The earliest work which deals with plastic rather than elastic effects appears to be that of Graebel (1961), who showed that a Reiner–Rivlin fluid is less stable than a Newtonian fluid. In a series of papers, Thomas & Walters (1963, 1964*a*, *b*) and Beard, Davis & Walters (1966) employed the Walters '*B*' fluid. Slightly elastic fluids were shown to be less stable, and highly elastic fluids even less stable, than a Newtonian fluid. Failure to find stationary instability for certain ranges of the parameters was attributed to the wave instability in these ranges; later this was indeed found to be the case. By contrast, employing the Walters '*A*' fluid, Chan Man Fong (1965) obtained increased stability, which suggests that the critical Taylor number may be quite sensitive to the rheological model considered for stability problems. Furthermore, Datta (1964) demonstrated, using a three-constant second-order Rivlin–Ericksen model, the instability to be greatly affected by the material constants of the rheological model. Along the same lines, Miller & Goddard (1967) employed the Coleman–Noll model with fading memory; they concluded that eight material functions are necessary to prescribe the fluid and six of these appreciably influence the problem; their experiments with a variety of polymer solutions showed these to be less stable than a Newtonian fluid. In the present study, we consider a convected Maxwell fluid defined by

$$\tau_{ij} + \lambda\, \delta\tau_{ij}/\delta t = 2\mu\epsilon_{ij}, \tag{1}$$

where τ_{ij} is the deviatoric part of the stress tensor, λ and μ are material constants, ϵ_{ij} is the strain-rate tensor and in terms of τ_{ij}, for example,

$$\frac{\delta\tau_{ij}}{\delta t} \equiv \delta_t \tau_{ij} \equiv \frac{D\tau_{ij}}{Dt} - \left(\tau_{ik}\frac{\partial V_j}{\partial x_k} + \tau_{kj}\frac{\partial V_i}{\partial x_k}\right).$$

This model, as demonstrated by Oldroyd (1950), combines simplicity and tensorial invariance with a certain ability to predict qualitatively the behaviour of viscoelastic fluids. Since we are primarily concerned with qualitative effects of elasticity on the stability of the slot problem, the present model appears to be adequate for our purpose. Clearly, the capacity of this model to predict rheological behaviour, compared with a model such as the one used by Miller & Goddard, is somewhat limited. However, since the stress-strain rates involved with buoyancy-driven flows are orders of magnitude smaller than those associated with forced flows, the use of this model for buoyancy-driven flows might be expected to yield more realistic results than those which would be obtained by its application to forced flows such as the Taylor problem.

2. Formulation

2.1. *General considerations and primary flow*

Consider the natural convection of a viscoelastic fluid in a vertical slot. Let the vertical walls of the slot be a distance d apart and be kept at uniform but different temperatures, say T_0 and T_1. For a Boussinesq fluid, neglecting the viscous dissipation (note that the elastic part of the fluid produces no dissipation), we have

$$\partial_t v_l + v_k \partial_k v_l = -(1/\rho_0)\,\partial_l p + \gamma g \theta \lambda_l + (1/\rho_0)\,\partial_j \tau_{jl}, \tag{2}$$

$$\partial_t T + v_k \partial_k T = \kappa \partial_k \partial_k T, \tag{3}$$

$$\tau_{kl} + t_0 \delta_t \tau_{kl} = \mu(\partial_l v_k + \partial_k v_l), \tag{4}$$

$$\partial_i v_i = 0, \tag{5}$$

where $\gamma = (\rho_0 - \rho)/\rho_0(T - T_m)$ is the coefficient of thermal expansion, ρ_0 the reference value of the density, p the excess pressure over the hydrostatic value, $\lambda_l = (1, 0, 0)$, $\theta = T - T_m$, $T_m = \frac{1}{2}(T_0 + T_1)$ and the other notation is conventional. The boundary conditions to be satisfied in this formulation are

$$\left.\begin{aligned} u(\pm\tfrac{1}{2}d) = v(\pm\tfrac{1}{2}d) = w(\pm\tfrac{1}{2}d) = 0,\\ T(-\tfrac{1}{2}d) = T_0, \quad T(+\tfrac{1}{2}d) = T_1. \end{aligned}\right\} \tag{6}$$

It can readily be shown that the rheological model selected has no effect on the primary flow corresponding to Newtonian fluids,

$$\hat{\Theta} = Y, \quad \hat{u} = \tfrac{1}{24}Y - \tfrac{1}{6}Y^3, \tag{7}$$

where $Y = y/d$, $\hat{\Theta} = (\bar{T} - T_m)/\Delta T$, $\Delta T = T_1 - T_0$, $\hat{u} = \bar{u}d/\nu G$, $G = g\gamma\Delta T d^3/\nu^2$, overbars designate dimensional properties of the base flow and G is the Grashof number.

2.2. *Stability problem*

Following the usual steps of linear stability theory, eliminating the shear stress, we obtain for the X momentum (in the direction of the gravitational field)

$$\begin{aligned} i\alpha\chi G[i\alpha(\hat{u} + c)\,U + VD\hat{u}] = {} & \alpha^2\chi GP + i\alpha\{D^2U + 2D\chi DU + UD^2\chi \\ & - [\alpha^2 + \beta^2 - 2(D\chi)^2]\,U\} + (D\chi/\chi) \\ & \times \{D^2V + 2D\chi DV + VD^2\chi - [\alpha^2 + \beta^2 - 2(D\chi)^2]\,V\} \\ & + 2(D^2\chi/\chi)\,DV + (D^3V - 2D\chi D^2\chi/\chi)\,V + i\alpha\chi T^+, \end{aligned} \tag{8}$$

for the Y momentum (perpendicular to the plates)

$$i\alpha\chi G(\hat{u}+c)\,V = -\chi GDP+\{D^2V+2D\chi DV+VD^2\chi-[\alpha^2+\beta^2-2(D\chi)^2]\,V\}, \quad (9)$$

for the Z momentum

$$i\alpha\chi G(\hat{u}+c)\,W = -i\beta\chi GP+\{D^2W+2D\chi DW+WD^2\chi-[\alpha^2+\beta^2-2(D\chi)^2]\,W\}, \quad (10)$$

for the thermal energy

$$(D^2-\alpha^2-\beta^2)\,T^+ = i\alpha R(\hat{u}+c)\,T^+ + RVD\hat{\Theta} \quad (11)$$

and for continuity $$i\alpha U+DV+i\beta W = 0, \quad (12)$$

with $$U(\pm\tfrac{1}{2}) = V(\pm\tfrac{1}{2}) = W(\pm\tfrac{1}{2}) = T^+(\pm\tfrac{1}{2}) = 0, \quad (13)$$

where $D \equiv d/dY$, α and β are the (real) wavenumbers in the X and Z directions, $c = c_r+ic_i$ is the (complex) wave speed, $X = x/d$, $Y = y/d$, $Z = z/d$, $t^+ = t\bar{U}_m/d$, $\bar{U}_m = \nu G/d$, $u' = u/\bar{U}_m$, $v' = v/\bar{U}_m$, $w' = w/\bar{U}_m$, $T' = (T-T_m)/\Delta T$, $p' = p/\rho_0\bar{U}_m^2$, $\Gamma = \nu t_0/d^2$, $R = G\sigma$, $\sigma = \nu/\kappa$,

$$\{u',v',w',p',T'\}(X,Y,Z,t^+) = \{U,V,W,P,T^+\}(\alpha,\beta,c;\,Y) \times \exp[i(\alpha X+\beta Z+\alpha ct^+)]$$

and $\chi = 1+i\alpha\Gamma G(\hat{u}+c)$. Here R denotes the Rayleigh number, σ the Prandtl number and $t_0 = \mu/G^*$, G^* being the modulus of shear rigidity.

It can be shown, after lengthy but straightforward calculations (see Gözüm 1972 for details), that Squire's (1933) theorem continues to hold for the present problem, and one need only consider two-dimensional disturbances.

Employing the present nomenclature for two-dimensional disturbances and for convenience, introducing $\Theta = -i\alpha T^+$ and eliminating P and U, we have

$$i\alpha G\chi[(\hat{u}+c)(D^2-\alpha^2)\,V-(D^2\hat{u})\,V]-\chi D\Theta = D^4V+M_3D^3V+M_2D^2V+M_1DV+M_0V, \quad (14)$$

$$(D^2-\alpha^2)\,\Theta = i\alpha R(\hat{u}+c)\,\Theta - i\alpha RVD\hat{\Theta}, \quad (15)$$

$$V(\pm\tfrac{1}{2}) = DV(\pm\tfrac{1}{2}) = \Theta(\pm\tfrac{1}{2}) = 0, \quad (16)$$

where

$$M_3 = 2(1-1/\chi)\,D\chi,$$
$$M_2 = -2\alpha^2+3(1-1/\chi)\,D^2\chi-(1-1/\chi)^2(D\chi)^2,$$
$$M_1 = -2\alpha(1-1/\chi)\,D\chi+2(1-1/\chi)\,D^3\chi+4(1-1/\chi)^2D\chi D^2\chi - 4(1/\chi-1/\chi^2)(D\chi)^3,$$
$$M_0 = \alpha^4+D^4\chi-\alpha^2(1-1/\chi)\,D^2\chi-2\alpha^2(1-1/\chi^2)(D\chi)^2-4(D\chi D^3\chi)/\chi - 3(D^2\chi)/\chi+4(D\chi)^4/\chi^2-(6/\chi)(1-1/\chi)\,D^2\chi(D\chi)^2.$$

Furthermore, because of the rheological model employed, the foregoing formulation is physically meaningful only when the effect of elasticity is small. Accordingly, assuming $G\Gamma \ll 1$, the system may be reduced to

$$(D^2-\alpha^2)^2\,V+i\alpha G\chi[VD^2\hat{u}-(\hat{u}+c)(D^2-\alpha^2)\,V]+\chi D\Theta = 0, \quad (17)$$

$$(D^2-\alpha^2)\,\Theta - i\alpha R[VD\hat{\Theta}-(\hat{u}+c)\,\Theta] = 0, \quad (18)$$

$$V(\pm\tfrac{1}{2}) = DV(\pm\tfrac{1}{2}) = \Theta(\pm\tfrac{1}{2}) = 0. \quad (19)$$

For the stability of viscoelastic Poiseuille flow, Chun & Schwarz (1968) previously obtained (17) with $\Theta \equiv 0$. Also, Mook (1971) recently demonstrated that (17) with $\Theta \equiv 0$ is the simplest possible equation showing the effect of elasticity on the Orr–Sommerfeld equation.

We proceed now to the solution of (17)–(19), which, with fixed σ, α, c and Γ, constitute an eigenvalue problem for G. The problem is not self-adjoint but, following Roberts (1960), it can be related to variational problems employing adjoint systems. (Gözüm (1972) gives a formulation of this nature; a convenient reference for the application of this concept to the formulation and solution of a variety of problems is Chandrasekhar (1961).) The use of the same concept for a numerical solution of the present problem, in view of the large number of variable coefficients involved in (17) and (18), is not practical. Here, we conveniently follow a procedure which combines the Galerkin method with the solution technique used for adjoint systems.

3. Method of solution

First, let (17) and (18) be rearranged as

$$(D^2-\alpha^2)^2 V = -i\alpha G\chi[VD^2\hat{u}-(\hat{u}+i\partial/\alpha\partial t^+)(D^2-\alpha^2)V]-\chi D\Theta, \qquad (17a)$$

$$(D^2-\alpha^2)\Theta = -i\alpha R[VD\hat{\vartheta}-(\hat{u}+i\partial/\alpha\partial t^+)\Theta], \qquad (18a)$$

subject to (19). In contrast to the application of the Galerkin procedure, which requires the selection of trial functions both for velocity and temperature, here we select trial functions only for the temperature. The form of (17*a*), (18*a*) and (7) suggests the set

$$\Theta = \sum_{n=1}^{\infty} a_n \cos\lambda_n^* Y + i\sum_{n=1}^{\infty} b_n \sin\mu_n Y, \qquad (20)$$

with $\lambda_n^* = \frac{1}{2}(2n-1)\pi$ and $\mu_n = 2\pi n$ $(n = 1, 2, 3, \ldots)$, where a_n and b_n are real functions of time.

Inserting (20) into (18*a*), solving (18*a*) for V and introducing this V and (20) into the right-hand side of (17*a*) reduces (17*a*) to a fourth-order, ordinary, nonhomogeneous differential equation with constant coefficients. Integration of this equation taking into consideration the appropriate boundary conditions gives a new V. Using this V and (20), we orthogonalize (18*a*) with respect to $\cos\lambda_n^* Y$ and $\sin\mu_n Y$. The resulting set has the general form

$$\mathbf{A}\, d\mathbf{X}/dt^+ = \mathbf{B}\mathbf{X}. \qquad (21)$$

The explicit forms of the matrices $\mathbf{A}$, $\mathbf{B}$ and $\mathbf{X}$ are rather lengthy, and are not given here (see Gözüm 1972 for details). If (20) is truncated after N terms, (21) represents $6N$ differential equations.

The eigenvalue problem is solved by finding the latent roots of the matrix $\mathbf{A}^{-1}\mathbf{B}$, when the onset of the instability is caused by travelling disturbances $(c_r \neq 0)$, and for stationary disturbances, by investigating the sign change of $\det\mathbf{B}$. The complete spectrum of eigenvalues is obtained by transforming the matrix $\mathbf{A}^{-1}\mathbf{B}$ to the upper almost triangular (Hessenberg) form, and then employing the QR algorithm (see, for example, Wilkinson 1965, p. 515). Subroutines for both these methods are available on the Michigan Terminal System. The accuracy of the eigenvalue subroutine is tested by computing the difference between tr $\mathbf{A}^{-1}\mathbf{B}$ and the sum of the eigenvalues. In all cases this difference was found to be less than 0·01 %. When an N-term approximation is used in (20), $\mathbf{A}^{-1}\mathbf{B}$ has $6N$ complex eigenvalues, say c_N ($N = 1, 2, 3, \ldots, 6N$). If the eigenvalues are rearranged such that $\mathrm{Im}\,(c_1) > \mathrm{Im}\,(c_2) > \ldots > \mathrm{Im}\,(c_{6N})$, then (17*a*), (18*a*) and (19) are neutrally stable when $\mathrm{Im}\,(c_1) = 0$, stable when $\mathrm{Im}\,(c_1) > 0$ and unstable when $\mathrm{Im}\,(c_1) < 0$. When $\mathrm{Re}\,(c_1) \neq 0$, unstable disturbances grow in the form of travelling waves; when $\mathrm{Re}\,(c_1) = 0$, unstable disturbances grow into stationary cells. For the latter case a necessary and sufficient condition is the vanishing of det $\mathbf{B}$. This approach decreases the computer time considerably, and is also used as a check during the evaluation of the eigenvalues of $\mathbf{A}^{-1}\mathbf{B}$.

4. Results and discussion

First, after eliminating the effect of elasticity from the formulation, the accuracy of the present method of solution was checked with the literature on Newtonian fluids. As was recently reported by Korpela, Gözüm & Baxi (1973), the instability in a slot sets in for $\sigma < 12{\cdot}7$ in the form of stationary cells and for $\sigma > 12{\cdot}7$ in the form of travelling waves. For $\sigma = 10^{-3}$, for example, the present method of solution yields results which agree to within 0·1 % with those obtained by Vest & Arpaci (1969*a*) for stationary disturbances. Beyond $\sigma = 12{\cdot}7$, it yields results which asymptotically approach those obtained by Gill & Kirkham (1970) for travelling waves corresponding to $\sigma \to \infty$; at $\sigma = 10^4$, for example, the present method and that of Gill & Kirkham give identical results for the wavenumber and the wave speed, while differing by 8 % on the critical Grashof number. Furthermore, the present method yields for travelling waves a critical Grashof number identical to that obtained by Korpela *et al.*, who followed the usual Galerkin approach. These comparisons increase confidence in the convergence of the present method of solution.

Now we proceed to viscoelastic results. A few checks with $N = 10$ show the results based on $N = 8$ to be accurate to within 2 %. As might be expected, the inherent complexity of the present problem makes *a priori* physical arguments difficult. From the results plotted in figures 1–4, we learn that slightly viscoelastic fluids slightly stabilize stationary disturbances, strongly destabilize wave disturbances, slightly decrease the critical wavenumber and strongly increase the critical wave speed. However, *a posteriori* arguments present no difficulty; first, in spite of well-known differences between the Bénard and slot problems for Newtonian fluids, elasticity appears to affect these problems in the same way, except for stationary disturbances. It does not affect the Bénard problem for these disturbances; this fact can readily be seen by inspecting the effect of Γ (and, consequently, that of χ) on the disturbance equations associated with the Bénard problem.

Second, the effect of Prandtl number on the present problem can easily be predicted; first of all, let $\sigma > 10$ because of the inherent nature of viscoelastic

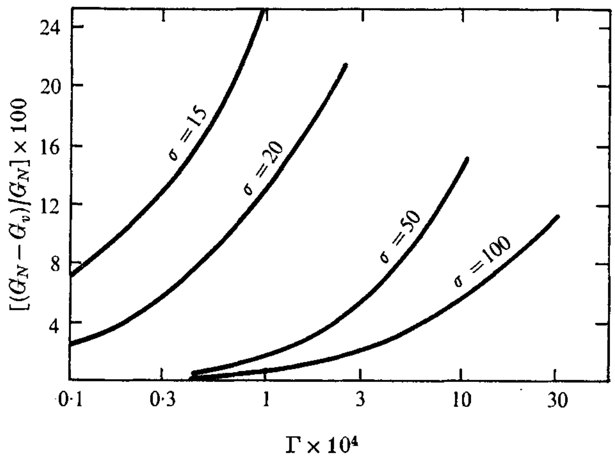

FIGURE 1. Strong destabilizing effect of elasticity for various Prandtl numbers. The subscripts N and v stand for 'Newtonian' and 'viscoelastic' respectively.

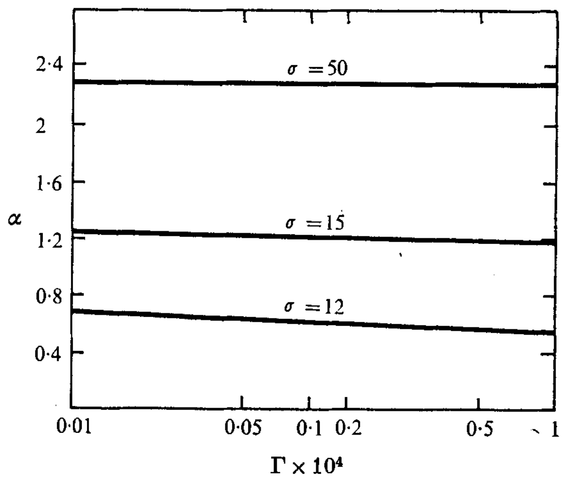

FIGURE 2. Slight effect of elasticity on the critical wavenumber for various Prandtl numbers.

fluids. Then, for an order-of-magnitude argument, we may use for $\sigma \to \infty$ and $\Gamma \to 0$ the result $G_c \sim 1/\sigma^{\frac{1}{2}}$ previously obtained by Gill & Kirkham (1970); consequently, in the marginal state, we have $\chi = 1 + O(\Gamma/\sigma^{\frac{1}{2}})$, which shows the effect of elasticity to decrease with increasing σ. Combining this result with the fact that elasticity has a slight effect on stationary disturbances but a strong one on travelling ones, we conclude that the maximum effect of elasticity should occur on the wave side of the transition between stationary cells and travelling waves, which is in the neighbourhood of $\sigma = 12$. This is why figure 4 is plotted for this value of the Prandtl number, and for reference, the numerical results used in this figure are tabulated in table 1.

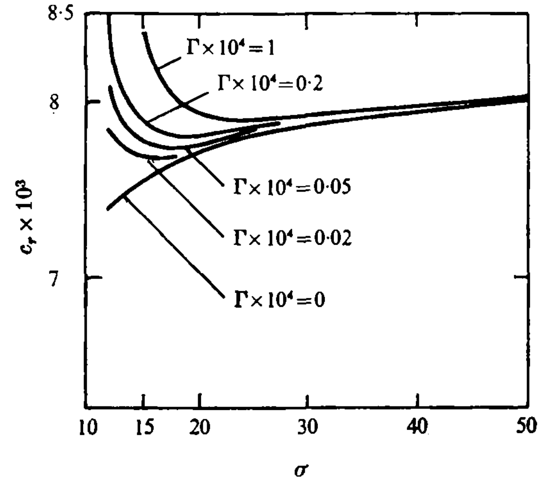

FIGURE 3. Strong effect of elasticity on the critical wave speed versus Prandtl number.

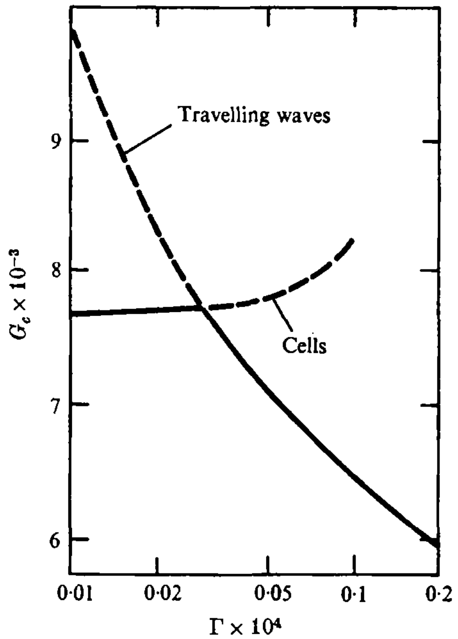

FIGURE 4. Effect of elasticity on the critical Grashof number for stationary and travelling disturbances. $\sigma = 12$.

5. Experiments

The goal of the experimental programme was a rather modest one; it was not to determine the adequacy of the Oldroyd '*B*' fluid for problems falling into the same category as the present one, nor was it to support *quantitatively* the preceding analytical results; the main interest was in finding *qualitative* experimental

$\Gamma \times 10^4$	α	G_c	$c_r \times 10^3$
0	2·65	7725	0
0·05	2·65	7800	0
0·1	2·65	8200	0
0·01	0·705	9800	7·3
0·02	0·675	8300	7·8
0·05	0·630	7125	8·1
0·2	0·600	6080	8·5

TABLE 1. Effect of elasticity on the critical Grashof number for stationary and travelling disturbances, $\sigma = 12$

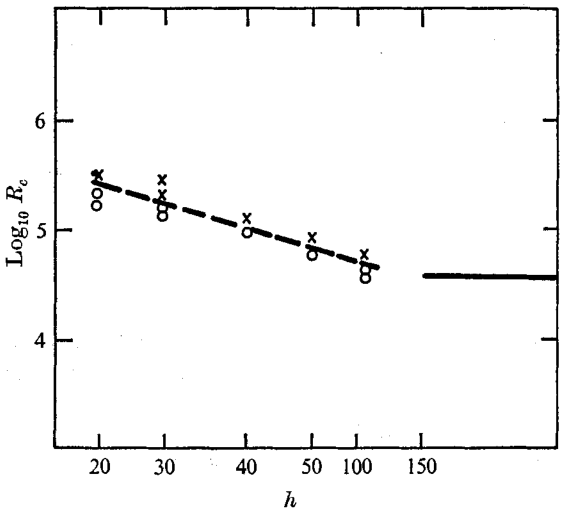

FIGURE 6. Measured critical Rayleigh number for various aspect ratios (Newtonian test fluid is silicone oil DC 200/2). $\sigma = 30$. ○, stable; ×, travelling waves; ——, analytical R_c for $h \to \infty$; – – –, experimental $R_c = f(h)$.

support for the fact that slightly viscoelastic fluids strongly destabilize wave disturbances, which is the major outcome of the present analytical study.

The apparatus, originally constructed by Vest (1967) to observe the onset of instability by flow visualization and temperature measurements, was modified by increasing the aspect ratio to a maximum of 105 so that the conduction regime could be approached. Experiments were carried out with a viscoelastic sample which was obtained by adding very small amounts of Separan AP30 to water. The viscosity of the sample was measured using a Brookfield Synchro-Lectric viscometer. At about 20 °C, the viscosity of the sample was found to be 4, 3·8 and 3·5 centipoise for speeds of 6, 12 and 30 r.p.m., respectively. Under the present shear-rate conditions, provided that the thermal conductivity, specific gravity and specific heat of the sample are close to those of water, $\sigma = 30$ is obtained after averaging. With the slot of 6 mm width, for the properties of the sample at about 20 °C, the Grashof number turns out to be $G = 16\Delta T$. Consequently, very large temperature differences are required to attain Grashof

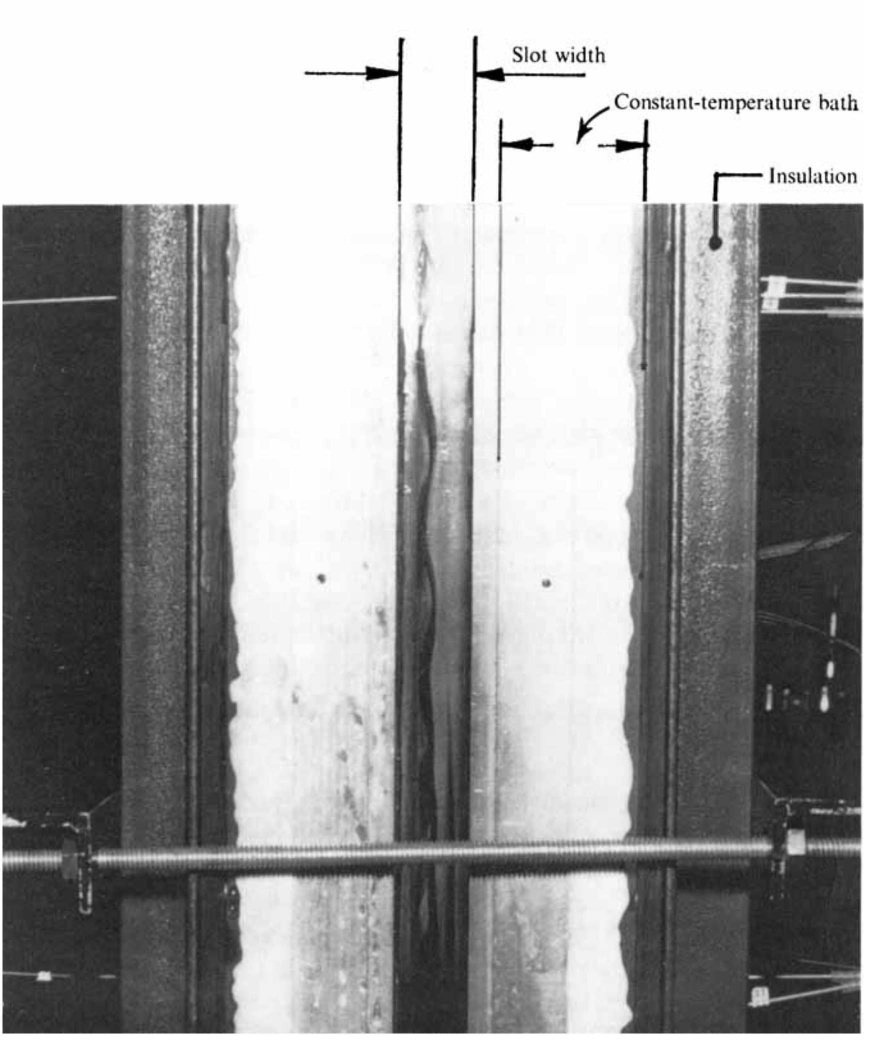

FIGURE 5. Wave instability of viscoelastic fluid ($G = 5500$, $\sigma = 30$, $h = 30$). Newtonian fluid is stable under same conditions.

numbers in the neighbourhood of 1000. Experiments up to temperature difference $\Delta T = 10\,^{\circ}\mathrm{C}$ showed no change in the stable flow. Since larger temperature differences cannot be justified because of the variation of fluid properties, an alternative was to increase the width of the slot so that small temperature differences start instability. For example, a slot of 2 cm width gives an aspect ratio of about 35 and appears to be suitable for the purpose. The flow pattern in this slot was visualized by injecting ink into the fluid. At $G = 3900$ the flow was stable while $G = 4300$ indicated wave motions. Figure 5 (plate 1) shows the waves at $G = 5500$. On the other hand, as seen from figure 6, a Newtonian fluid (silicon oil DC 200/2) with the same Prandtl number remains stable until $G = 6000$.

Dr Gözüm was killed in a car accident in May 1972, and this paper has been extracted from his Ph.D. thesis by V. S. A. Thanks are expressed to Prof. C. M. Vest and Prof. S. A. Korpela for their remarks on an earlier version of the paper.

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